JOURNAL OF APPROXIMATION THEORY 19, 231–238 (1977)

Rational Approximation to $|x|/1 + x^{2m}$ on $(-\infty, +\infty)$

DONALD J. NEWMAN

Department of Mathematics, Temple University, Philadelphia, Pennsylvania 19122

AND

A. R. REDDY

School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540

Communicated by Oved Shisha

Received September 19, 1975

Freud [2] has obtained the following theorems by using a well-known result of Newman [4].

THEOREM 1. *There is a rational function $r_n^*(x) = P_n(x)/Q_n(x)$ of degree at most n, for which for all $n \geqslant 5$,*

$$\left\| \frac{|x|}{1+x^2} - r_n^*(x) \right\|_{L_\infty(-\infty,+\infty)} \leqslant e^{-C_1 n^{1/2}}.$$

THEOREM 2. *For every rational function $r_n(x)$ of degree at most n, we have for all $n \geqslant 5$,*

$$\left\| \frac{|x|}{1+x^2} - r_n(x) \right\|_{L_\infty(-\infty,+\infty)} \geqslant e^{-C_2 n^{1/2}}.$$

Theorems 3 and 4 of this note are results on the approximation of $|x|/1 + x^{2m}$ by reciprocals of polynomials of degree $\leqslant n$ on $(-\infty, +\infty)$, m being a positive integer. Theorems 5 and 6 show that the exact order of approximability of $x/1 + x^{2m}$ on $[0, \infty)$ by reciprocals of polynomials of degree $\leqslant n$ $Cn^{(1/m)-2}$. Finally we show that $(1+x)/(1+x^2)$ can be approximated by reciprocals of polynomials of degree $\leqslant n$ on $[0, \infty)$ to the order of $C_5(\log n)n^{-1}$ but not better than $C_6 n^{-6}$.

We use throughout our work $C, C_1, C_2, C_3, \ldots$, to denote positive constants and $T_n(x)$ to denote the nth Chebyshev polynomial of the firstkind.

ISSN 0021–9045

LEMMAS

LEMMA 1. *There is a polynomial $P_n^*(x)$ of degree $\leqslant n$ for which*

$$\left\| |x| - \frac{1}{P_n^*(x)} \right\|_{L_\infty[-1,1]} \leqslant \frac{C_1}{n}.$$

Remark. This result improves a recent result of Lungu [3].

Proof. Let n be odd and

$$P(x) = \frac{1}{Cx} \int_0^x \left(\frac{T_n(t)}{t}\right)^2 dt;$$

C is chosen so that $P^*(1) = 1$.

Clearly $P(x)$ is an even polynomial of degree $2n - 2$. By evenness we consider only $x \in [0, 1]$. Write

$$\frac{1}{P(x)} - x = \frac{x \int_x^1 \left(\frac{T_n(t)}{t}\right)^2 dt}{\int_0^x \left(\frac{T_n(t)}{t}\right)^2 dt}.$$

Now if we write $t = \sin\theta$, then $T_n(t) = (-1)^{(n-1)/2} \sin n\theta$ and so we have the upper bounds, $|T_n(t)/t| \leqslant n$, $|T_n(t)/t| \leqslant 1/t$, and a lower bound, $|T_n(t)/t| \geqslant 2/\pi\, n$, throughout $0 \leqslant t \leqslant \sin \pi/2n$.

For $0 \leqslant x \leqslant \sin \pi/2n$, $\int_0^x (T_n(t)/t)^2\, dt \geqslant 4n^2x/\pi^2$;

$$\int_x^1 \left(\frac{T_n(t)}{t}\right)^2 dt \leqslant \int_0^{1/n} \left(\frac{T_n(t)}{t}\right)^2 dt + \int_{1/n}^1 \left(\frac{T_n(t)}{t}\right)^2 dt$$
$$\leqslant \int_0^{1/n} n^2\, dt + \int_{1/n}^\infty \frac{dt}{t^2} = 2n.$$

Therefore, for $0 \leqslant x \leqslant \sin \pi/2n$,

$$\left| x - \frac{1}{P(x)} \right| \leqslant \frac{(2nx)\,\pi^2}{4n^2x} = \frac{\pi^2}{2n}. \tag{1}$$

For $\sin \pi/2n \leqslant x \leqslant 1$,

$$\int_0^x \left(\frac{T_n(t)}{t}\right)^2 dt \geqslant \int_0^{\sin(\pi/2n)} \left(\frac{T_n(t)}{t}\right)^2 dt \geqslant \frac{4n^2}{\pi^2} \sin(\pi/2n) \geqslant 4n/\pi^2,$$

$$\int_x^1 \left(\frac{T_n(t)}{t}\right)^2 dt \leqslant \int_x^\infty \frac{dt}{t^2} = \frac{1}{x},$$

and hence,

$$\left| x - \frac{1}{P(x)} \right| \leqslant \frac{x(1/x)}{4n/\pi^2} = \frac{\pi^2}{4n}. \tag{2}$$

The lemma follows easily from (1) and (2).

LEMMA 2. [1, Lemma 5]. *There exists a sequence* $\{P_n(x)\}_{n=1}^{\infty}$ *of polynomials of degree* $\leqslant n$ *for which*

$$\left\| x - \frac{1}{P_n(x)} \right\|_{L_\infty[0,1]} \leqslant C_4 n^{-2}. \tag{3}$$

Proof. Choose n even and set

$$Q_n(x) = \frac{(\cos \pi/2n - \cos \pi/n)}{x + \cos \pi/2n - \cos \pi/n} T_n(x - \cos \pi/n).$$

This is a polynomial since $T_n(-\cos \pi/2n) = 0$. Since $Q(0) = 1$, $P_n(x) = (1 - Q_n(x))/x$ is a polynomial. Set $\delta = \cos \pi/2n - \cos \pi/n$: then on $[0, 1]$

$$\begin{aligned} \delta + x - \frac{1}{P_n(x)} &= \frac{\delta - (\delta + x)\, Q(x)}{1 - Q(x)} = \delta \frac{1 + (-1)^n T_n(x - \cos \pi/n)}{1 + \dfrac{(-1)^n}{1 + x/\delta} T_n(x - \cos \pi/n)} \\ &= M = \delta \frac{1 + t}{1 + st}, \qquad \text{where} \quad t \in [-1, 1], \quad s \in [0, 1]. \end{aligned} \tag{4}$$

Now it is easy to check that $0 \leqslant M \leqslant 2\delta$. Hence $0 \leqslant \delta + x - 1/P_n(x) \leqslant 2\delta$, i.e.,

$$\left| x - \frac{1}{P_n(x)} \right| \leqslant \delta < \frac{\pi^2}{2n^2}.$$

Hence, the lemma is proved.

LEMMA 3. [6, p. 68]. *Let* $P(x)$ *be any polynomial of degree at most* n *satisfying* $|P(x)| \leqslant M$ *on* $[a, b]$. *Then at any point outside* $[a, b]$, *we have*

$$|P(x)| \leqslant M \left| T_n\left(\frac{2x - a - b}{b - a}\right)\right|.$$

NEW THEOREMS

THEOREM 3. *There is a polynomial* $P_n^*(x)$ *of degree* $\leqslant n$ *for which for all large* n ($n > 2m$, *where* m *is any fixed positive integer*),

$$\left\| \frac{|x|}{1 + x^{2m}} - \frac{1}{P_n^*(x)} \right\|_{L_\infty(-\infty, \infty)} \leqslant \frac{C_9}{(n - 2m)^{1-1/2m}}. \tag{5}$$

Proof. By Lemma 1, we have for $0 \leqslant x \leqslant (n-2m)^{1/2m}$, and an appropriate polynomial P_{n-2m} of degree $\leqslant n-2m$,

$$\left\| |x| - \frac{1}{P_{n-2m}(x)} \right\| \leqslant \frac{C_{10}}{(n-2m)^{1-1/2m}}.$$

Therefore, for $0 \leqslant |x| \leqslant (n-2m)^{1/2m}$,

$$\left| \frac{|x|}{1+x^{2m}} - \frac{1}{(1+x^{2m})\, P_{n-2m}(x)} \right| \leqslant \left| |x| - \frac{1}{P_{n-2m}(x)} \right| \leqslant \frac{C_{10}}{(n-2m)^{1-1/2m}}. \tag{6}$$

On the other hand, for $|x| > (n-2m)^{1/2m}$,

$$\left| \frac{|x|}{1+x^{2m}} - \frac{1}{(1+x^{2m})\, P_{n-2m}(x)} \right| \leqslant \frac{2|x|}{1+x^{2m}} \leqslant \frac{C_{11}}{|x|^{2m-1}} < \frac{C_{11}}{(n-2m)^{1-1/2m}}, \tag{7}$$

since from the construction of the polynomial $P(x)$ in Lemma 1, we have for all $|x| \geqslant 1$,

$$\left| \frac{1}{P(x)} \right| \leqslant |x|.$$

Now (5) follows from (6) and (7).

THEOREM 4. *For e very polynomial $P_n(x)$ of degree $\leqslant n$,*

$$\left\| \frac{x^{1/2}}{1+x^m} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \geqslant \frac{1}{20n^{1-1/2m}}. \tag{8}$$

Remark. By changing $x^{1/2}$ to x in (8), we get

$$\left\| \frac{|x|}{1+x^{2m}} - \frac{1}{P_n(x^2)} \right\|_{L_\infty[-\infty,\infty)} \geqslant \frac{C_{12}}{n^{1-1/2m}}. \tag{9}$$

Proof. Let us assume on the contrary

$$\left\| \frac{x^{1/2}}{1+x^m} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} < \frac{1}{20n^{1-1/2m}}. \tag{10}$$

From (10) we obtain for $\alpha_n = (12n^{2-1/m})^{-1} \leqslant x \leqslant n^{1/m}$,

$$\frac{1}{P_n(x)} \geqslant \frac{x^{1/2}}{1 + x^m} - \frac{1}{20n^{1-1/2m}} \geqslant \frac{n^{1/2m}}{4n} - \frac{1}{20n^{1-1/2m}} = \frac{n^{1/2m}}{5n}, \quad (11)$$

i.e.,

$$\operatorname*{Max}_{\alpha_n \leqslant x \leqslant n^{1/2m}} |P_n(x)| \leqslant 5n^{1-1/2m}. \quad (12)$$

Hence by applying Lemma 3 to (12) we get

$$|P_n(0)| \leqslant 14n^{1-1/2m}. \quad (13)$$

On the other hand we get by (10),

$$\left| \frac{1}{P_n(0)} \right| < \frac{1}{20n^{1-1/2m}}, \quad (14)$$

which is inconsistent with (13).

THEOREM 5. *For every polynomial $P_n(x)$ of degree $\leqslant n$ and for all $n \geqslant 2$,*

$$\left\| \frac{x}{1 + x^{2m}} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \geqslant \frac{1}{(32n^{1-1/2m})^2}. \quad (15)$$

Proof. Suppose

$$\left\| \frac{x}{1 + x^{2m}} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} < \frac{1}{(32n^{1-1/2m})^2}. \quad (16)$$

From (16) we obtain, for $\beta_n = (n^{2-1/m})^{-1} \leqslant x \leqslant n^{1/m}$,

$$\frac{1}{P_n(x)} > \frac{x}{1 + x^{2m}} - \frac{1}{(32n^{1-1/2m})^2} \geqslant \frac{1}{(2n^{1-1/2m})^2} - \frac{1}{(32n^{1-1/2m})^2} > \frac{210}{(32n^{1-1/2m})^2};$$

hence,

$$\operatorname*{Max}_{\beta_n \leqslant x \leqslant n^{1/m}} |P_n(x)| < \frac{(32n^{1-1/2m})^2}{210}. \quad (17)$$

By applying Lemma 3 to (17) we get

$$|P_n(0)| \leqslant 512(n^{1-1/2m})^2. \quad (18)$$

On the other hand, we have from (16),

$$|P_n(0)| > (32n^{1-1/2m})^2, \tag{19}$$

contradicting (18).

THEOREM 6. *There is a polynomial $Q_n^*(x)$ of degree at most n for which, for all large n ($n > 2m$, m is any fixed positive integer),*

$$\left| \frac{x}{1+x^{2m}} - \frac{1}{Q_n^*(x)} \right|_{L_\infty[0,\infty)} \leqslant C_{13}(n-2m)^{1/m-2}. \tag{20}$$

Proof. By Lemma 2, we have for $0 \leqslant x \leqslant (n-2m)^{1/m} = \delta_n$,

$$\left| x - \frac{\delta_n}{P_{n-2m}(x/\delta_n)} \right| \leqslant \frac{C_{14}(n-2m)^{1/m}}{(n-2m)^2}. \tag{21}$$

From the construction of the polynomial $P_n(x)$ in Lemma 2, we have

$$0 < 1/P_n(x) < 2x \qquad \text{for all} \quad x \geqslant 1,\ n \geqslant 4. \tag{22}$$

For $0 \leqslant x \leqslant \delta_n$, with $Q_n(x) \equiv \delta_n^{-1} P_{n-2m}(x/\delta_n)$,

$$\left| \frac{x}{1+x^{2m}} - \frac{1}{(1+x^{2m})\,Q_n(x)} \right| \leqslant \frac{C_{15}(n-2m)^{1/m}}{(n-2m)^2}. \tag{23}$$

On the other hand, for $x > (n-2m)^{1/m}$, we get for all large n, by (22),

$$\left| \frac{x}{1+x^{2m}} - \frac{1}{(1+x^{2m})\,Q_n(x)} \right| < \frac{3x}{1+x^{2m}} \leqslant \frac{1}{C_{16}x^{2m-1}}$$
$$< C_{17}(n-2m)^{-2+1/m}. \tag{24}$$

The required result (20) follows from (23) and (24).

THEOREM 7. *For every polynomial $P_n(x)$ of degree $n \geqslant 2$,*

$$\left\| \frac{1+x}{1+x^2} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \geqslant \frac{1}{120n^6}. \tag{25}$$

Proof. Suppose

$$\left\| \frac{1+x}{1+x^2} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} < \frac{1}{120n^6}. \tag{26}$$

From (26) we get, for $0 \leqslant x \leqslant n^2$,

$$\frac{1}{P_n(x)} \geqslant \frac{1+x}{1+x^2} - \frac{1}{120n^6} \geqslant \frac{\frac{3}{4}+x}{1+x^2} - \frac{1}{120n^6} \geqslant \frac{1}{2(1+x)} \geqslant \frac{1}{2(1+n^2)}. \tag{27}$$

Therefore, for $0 \leqslant x \leqslant n^2$,

$$|P_n(x)| < 2(1+n^2). \tag{28}$$

Set $Q_{n+1}(x) = (1+x)P_n(x) - (1+x^2)$. Then from (26) and (28), for $0 \leqslant x \leqslant n^2$,

$$|Q_{n+1}(x)| \leqslant \frac{(1+x^2)P_n(x)}{120n^6} < \frac{1}{45}. \tag{29}$$

Now by applying Lemma 3 to (29), we get

$$2 = |Q_{n+1}(-1)| < 2. \tag{30}$$

THEOREM 8. *There is a polynomial $P_n^*(x)$ of degree at most n for which*

$$\left\| \frac{1+x}{1+x^2} - \frac{1}{P_n^*(x)} \right\|_{L_\infty[0,\infty)} \leqslant \frac{C_{18}\log n}{n}. \tag{31}$$

Proof. Set $P_1(x) = 1+x^2$, and for n odd:

$$P_{2,n}(x) = T_n\left(\frac{9x\log n}{n} - 1\right).$$

$$P_n^*(x) = \frac{P_1(x) + CP_{2,n}(x)}{1+x};$$

C is such that $P(x)$ is a polynomial:

$$C = P_1(-1)\left[T_n\left(\frac{9\log n}{n} + 1\right)\right]^{-1} \leqslant n^{-9}.$$

Then on $[0, 2n\,(9\log n)^{-1}]$,

$$\begin{aligned}\left|\frac{1+x}{1+x^2} - \frac{1}{P_n^*(x)}\right| &= \left|\frac{1+x}{1+x^2} - \frac{1+x}{P_1(x)+CP_{2,n}(x)}\right| \\ &= (1+x)\left|\frac{1}{1+x^2} - \frac{1}{(1+x^2)+CP_{2,n}(x)}\right| \qquad (32)\\ &= \frac{(1+x)}{(1+x^2)}\left|\frac{CP_2(x)}{(1+x^2)+CP_{2,n}(x)}\right| \leqslant 2Cn \leqslant 2n^{-2}.\end{aligned}$$

On the other hand, for $x > 2n/(9 \log n)$,

$$\left|\frac{1+x}{1+x^2} - \frac{1}{P^*(x)}\right| \leqslant \frac{(1+x)}{(1+x^2)} \left|\frac{CP_{2,n}(x)}{(1+x^2)+CP_{2,n}(x)}\right| \leqslant \frac{1+x}{1+x^2} < \frac{2}{x} < C_{19}\frac{\log n}{n}. \tag{33}$$

The required result (31) follows from (32) and (33).

Note Added in Proof. By replacing $(\log n)n$ by $[(\log n)/n]^2$ and n^{-9} by n^{-3} in the proof of Theorem 8, one can replace $(\log n)/n$ in (31) by $[(\log n)/n]^2$.

References

1. Paul Erdös, Donald J. Newman, and A. R. Reddy, Rational approximation (II), *Advances in Math.*, to appear.
2. G. Freud, A remark concerning the rational approximation to $|x|$, *Studia Sci. Math. Hungar.* **2** (1967), 115–117.
3. K. N. Lungu, Best approximation of $|x|$ by rational functions of the form $1/P_n(x)$, *Siberian Math. J.* **15** (1974), 1152–1156.
4. D. J. Newman, Rational approximation to $|x|$, *Michigan Math. J.* **11** (1964), 11–14.
5. D. J. Newman and A. R. Reddy, Rational approximation (III), submitted for publication.
6. A. F. Timan, Theory of approximation of functions of a real variable, Macmillan, New York, 1963.